MATHEMATICS FOR MACHINE LEARNING

LINEAR ALGEBRA

SYSTEMS OF SENTENCES

eg: Goal: You are given info about the animals, you have to figure out the colour of the animal

So, our goal is to convey as much information as possible with these simple sentences.

System 1:

The dog is black
The cat is orange

} complete system ← This also called as non-singular system.

each sentence gives one piece of information.

System 2:

The dog is black
The dog is black

} redundant system

Both sentences give same piece of information.

System 3:

The dog is black
The dog is white

} contradictory system

Both sentences give contradictory information

These two are also called as singular system

eg: Between the dog, the cat, and the bird, one is red.
Between the dog and the cat, one is orange.
The dog is black.

These three sentences form a complete system aka. non-singular system, as all three sentences give one piece of information.

We can thus figure out that the bird is red.

SYSTEM OF EQUATIONS

SYSTEM OF SENTENCES ⟶ SYSTEM OF EQUATIONS.

eg: You bought an apple and a banana and they cost \$10
You bought an apple and two bananas and they cost \$12.

↓

$$\left.\begin{aligned} \text{apple} + \text{banana} &= 10 \\ \text{apple} + 2(\text{banana}) &= 12 \end{aligned}\right\}$$

← complete non-singular system

we can thus find the cost of each fruit from these two pieces of information. apple = \$8 ; banana = \$2

eg : You bought an apple and a banana for \$10
You bought two apples and two bananas for \$20.
Q : How much does each fruit cost ?
A : There is not enough information.

$$\left.\begin{aligned} \text{apple} + \text{banana} &= 10 \\ 2(\text{apple}) + 2(\text{banana}) &= 20 \end{aligned}\right\} \leftarrow \text{redundant system} \quad (\text{singular system})$$

ie. they have infinitely many solutions :
apple can cost 0, 8, 5, etc
bananca can cost 10, 2, 5, etc.

eg : You bought an apple and a banana and they cost \$10
you bought two apples and two bananas and they cost \$24.
Q : How much does each fruit cost ?
A : wrong information.

$$\left.\begin{aligned} \text{apple} + \text{banana} &= 10 \\ 2(\text{apple}) + 2(\text{banana}) &= 24 \end{aligned}\right\} \leftarrow \text{contradictory system} \quad (\text{singular system})$$

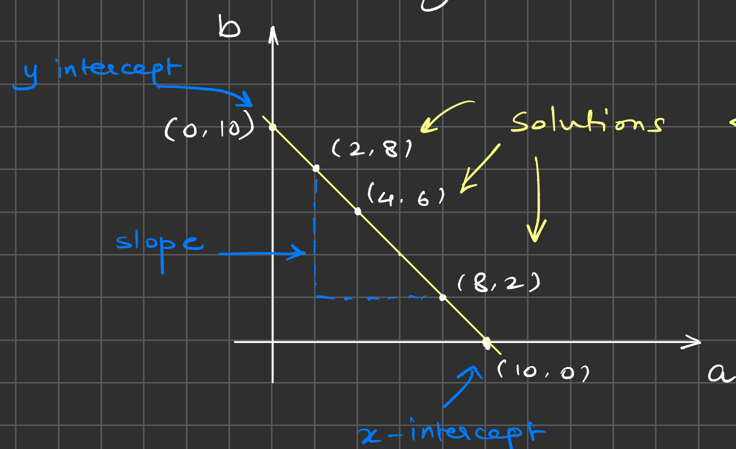
ie. they have <u>no</u> solutions.

Two types of equations :
- Linear equations — eg. $a + b = 10$; $2a + 3b - 4d + 5e = 72$, etc
- Non-linear equations — eg. $a^2 + b^2 = 10$; $\sin(a) + b^5 = 15$, etc.

Linear Algebra only focuses on systems of linear equations.

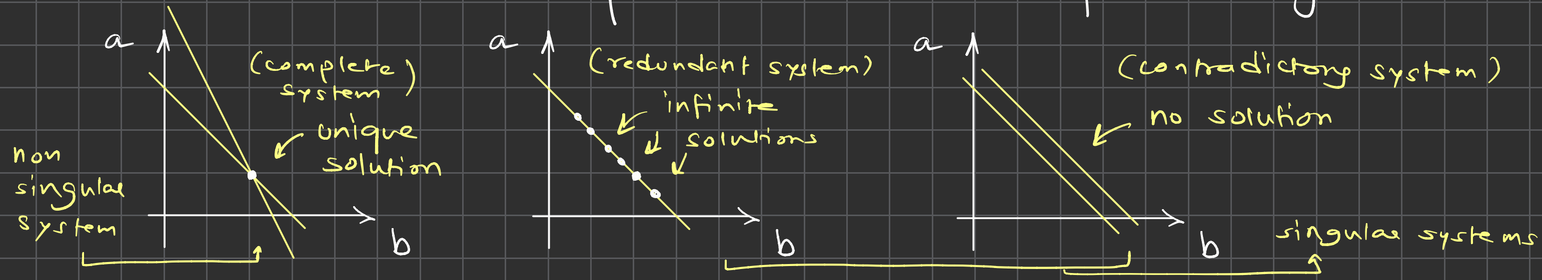
<u>LINEAR EQUATIONS ⟶ LINES ON A GRAPH.</u>

$a + b = 10$ (single line on a graph :)



⟸ Infinite solutions on the line.

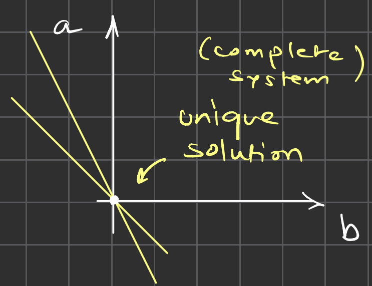
when two lines are plotted on the same plane we get,



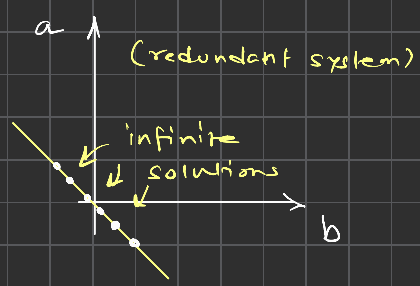
SINGULAR VS NON-SINGULAR SYSTEM:

If we put the constants of the linear equation systems to be zero, a non-singular system will still have a unique solution, whereas a redundant and contradictory system will have infinite solutions.

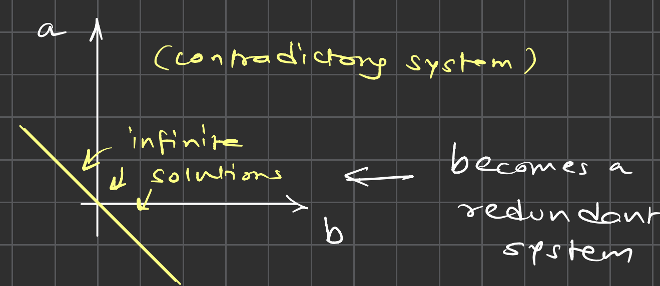
eg. Taking the previous graphs, and removing the constants, we get the following graphs:



non-singular system



singular system



singular system

(All of these go through the origin)
A non-singular system has only solution ⇒ the origin whereas a singular system has infinite solutions, the origin being one of them.

LINEAR EQUATIONS → MATRICES

eg.

$$a + b = 0$$
$$a + 2b = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Non-singular system
⇒ Non-singular matrix

(unique solution)

$$a + b = 0$$
$$2a + 2b = 0$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

singular system
⇒ singular matrix

(Infinitely many solutions)

LINEAR INDEPENDENCE

ex: $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

No row is a multiple of other one

∴ Rows are linearly independent

LINEAR DEPENDENCE

ex: $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Second row is a multiple of the first row

∴ Rows are linearly dependent

DETERMINANT

A matrix is singular if its determinant is equal to zero.
For a singular matrix, the rows are linearly dependent
$\therefore$ If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix,
we can say that,

$$[a \quad b] \times k = [c \quad d]$$

ie. $ak = c$; $bk = d$

$\therefore \frac{a}{c} = k$; $\frac{b}{d} = k$

$\therefore \frac{a}{c} = \frac{b}{d} = k$

$\therefore ad = bc$

$\Rightarrow \underline{\underline{ad - bc}} = 0$

↑ This is called the determinant of the matrix.

If the determinant is non-zero, then the matrix is non-singular.

(Q) Find determinant of $\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$

(A) Determinant:
$= 5 \times 3 - (-1)(1)$
$= 15 + 1$
$= 16$
$\therefore$ Non-singular matrix

(Q) Find determinant of $\begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$

(A) Determinant:
$= (2)(3) - (-1)(-6)$
$= 6 - 6$
$= 0$
$\therefore$ Singular matrix.

3 VARIABLE EQUATIONS

Similar to the two variable equations discussed earlier, three variable equations also have singular and non-singular system.

These linear equations form a 2-Dimensional plane in the 3-Dimensional space.

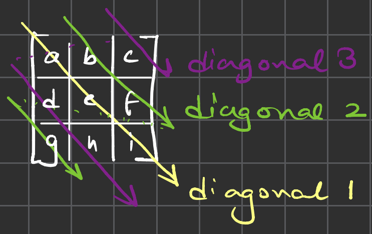
These linear equations form 3×3 matrices.

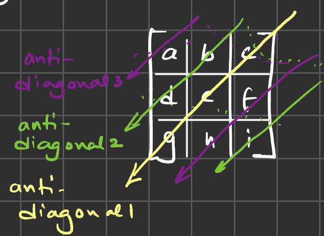
And, if there exists a linear relation among the rows of the matrix, we can say that the rows are linearly dependent, and thus a singular matrix.

If there is no relation among the rows of the matrix, we can say that the rows are linearly independent, and thus form a non-singular matrix.

DETERMINANT OF 3X3 MATRIX

In a 3×3 matrix the diagonals are shown as,





∴ Determinant =

$$(aei + bfg + cdh) - (ceg + bdi + afh)$$

We can also write the determinant as,

$$a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

this is the notation for a determinant

when it is a triangular matrix, the determinant is the product of the diagonals;

triangular matrix

$$\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix}$$

∴ determinant $= (a)(e)(i)$

SOLVING SYSTEM OF EQUATIONS

SYSTEM OF EQUATIONS ⟶ SOLVED SYSTEM

$a + b = 10$ $\qquad$ $a = 8$

$a + 2b = 12$ $\qquad$ $b = 2$

we manipulate the equations to reach the solved state.

MANIPULATING METHODS :

* Multiplying by a constant

$$\begin{array}{rl} a + b = & 10 \\ \times & 7 \\ \hline 7a + 7b = & 70 \end{array}$$

* Adding two equations

$$\begin{array}{r} a + b = 10 \\ 2a + 3b = 22 \\ \hline 3a + 4b = 32 \end{array}$$

SOLVING A SYSTEM OF EQUATIONS WITH 3 VARIABLES

We solve it using elimination method.

$$a + b + 2c = 12$$
$$3a - 3b - c = 3$$
$$2a - b + 6c = 24$$

(1) First divide each row by the coefficient of 'a'.
(2) Now, use the first equation to remove 'a' from the others
(3) We solve the other two equations to get the values of 'b' and 'c'.
(4) We plug those values in the first equation to get the value of 'a'.

MATRIX ROW REDUCTION (GAUSSIAN ELIMINATION)

original system ⟶ Intermediate system ⟶ Solved system

$$\begin{aligned} 5a + b &= 17 \\ 4a - 3b &= 6 \end{aligned} \qquad \begin{aligned} a + 0.2b &= 3.4 \\ b &= 2 \end{aligned} \qquad \begin{aligned} a + 0b &= 3 \\ 0a + b &= 2 \end{aligned}$$

original matrix ⟶ upper diagonal matrix ⟶ Diagonal matrix

$$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Row echelon form — Reduced Row echelon form

ROW ECHELON FORM

$$\begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

any number allowed

if the diagonal has a 0, these values need to be 0.

diagonal can be 0s or 1s.

this triangle has to be all 0s

eg: $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

All three are valid row echelon forms.

MATRIX ROW OPERATIONS

Matrix row operations preserve the singularity or the non singularity of a matrix.

* Switching Rows:

original $\longrightarrow$ switched

$$\begin{bmatrix} 5 & 1 \\ 4 & 3 \end{bmatrix} \qquad \begin{bmatrix} 4 & 3 \\ 5 & 1 \end{bmatrix}$$

Determinant = 11 Determinant = −11

Non-singularity or singularity is preserved even after switching rows.

The determinant of the updated matrix is the negative of the determinant of the original one.

* Multiplying a row by a (non-zero) scalar

original $\longrightarrow$ updated

$$\begin{bmatrix} 5 & 1 \\ 4 & 3 \end{bmatrix} \qquad \begin{bmatrix} 50 & 10 \\ 4 & 3 \end{bmatrix}$$

Determinant = 11 Determinant = 110

Non-singularity or singularity is preserved after updating rows.

The determinant of the updated matrix is the product of the scalar multiplier and the determinant of the original matrix.

* Adding a row to another row

original $\longrightarrow$ updated

$$\begin{bmatrix} 5 & 1 \\ 4 & 3 \end{bmatrix} \qquad \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

Determinant = 11 Determinant = 11

Non-singularity or singularity is preserved after updating rows.

The determinant of the updated matrix is same as the original matrix.

RANK OF A MATRIX

Application :
compression of images = reducing the rank

systems of Information :

system 1	System 2	system 3
The dog is black The cat is orange	The dog is black The dog is black	The dog The dog
Two sentences	Two sentences	Two sentences
Two pieces of information	one piece of information	zero pieces of information
∴ Rank = 2	∴ Rank = 1	∴ Rank = 0

∴ Rank is defined as the amount of information the system carries.

Systems of equations :

System 1	System 2	System 3
$a+b=x$ $a+2b=y$	$a+b=x$ $2a+2b=y$	$0a+0b=0$ $0a+0b=0$
Two equations	Two equations	Two equations
Two pieces of info.	one piece of info.	zero pieces of info.
∴ Rank = 2	∴ Rank = 1	∴ Rank = 0

∴ Converting these equations to matrices, we get

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

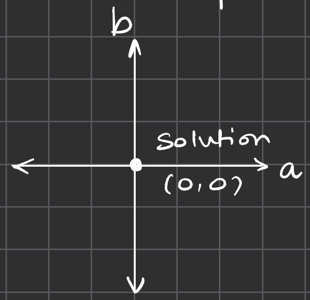
∴ rank of matrix = 2 ∴ rank of matrix = 1 ∴ rank of matrix = 0

RANK AND SOLUTIONS TO THE SYSTEM

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Rank = 2

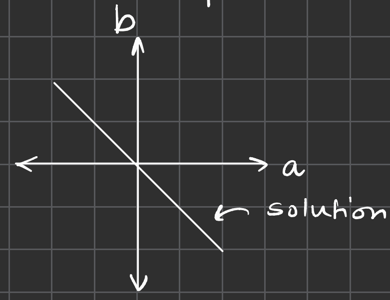
$\therefore$ Dimension of the solution space = 0



(A point on a graph has 0 dimensions)

Rank = 1

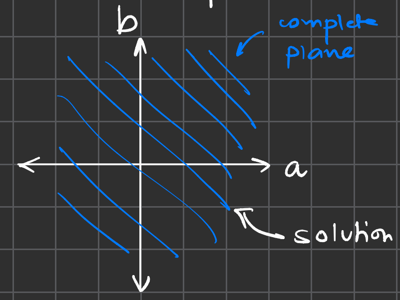
$\therefore$ Dimension of the solution space = 1



(A line on a graph has 1 dimension)

Rank = 0

$\therefore$ Dimension of the solution space = 2



(A plane has 2 dimensions)

$\therefore$ Rank $=$ 2 $-$ (Dimension of solution space).

$\uparrow$ # rows of matrix

A matrix is non-singular if it has full rank.

Full rank = # rows of the matrix.

A matrix is singular if it does not have the full rank.

For 3 x 3 matrices :

$a + b + c = 0$
$a + 2b + c = 0$
$a + b + 2c = 0$

3 equations
3 pieces of info.

$\therefore$ Rank = 3

$a + b + c = 0$
$a + b + 2c = 0$ $\leftarrow$ average of other two rows
$a + b + 3c = 0$

3 equations
2 pieces of info.

$\therefore$ Rank = 2

$a + b + c = 0$
$2a + 2b + 2c = 0$
$3a + 3b + 3c = 0$

3 equations
1 piece of info

$\therefore$ Rank = 1.

$0a + 0b + 0c = 0$
$0a + 0b + 0c = 0$
$0a + 0b + 0c = 0$

3 equations
0 pieces of info.
$\therefore$ Rank = 0

CALCULATING RANK OF A MATRIX

We get the row echelon form of the matrix by manipulating the rows.

Then the rank of the matrix is the number of 1s in the diagonal of the row echelon form.

eg. $\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$ # 1s in diagonal = 2
$\therefore$ Rank = 2

$\begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix}$ # 1s in diagonal = 1
$\therefore$ Rank = 1

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ # 1s in diagonal = 0
$\therefore$ Rank = 0

A matrix is said to be non-singular if the row echelon form diagonal has only 1s and no zeros.

ROW ECHELON FORM (contd.)

eg. $\begin{bmatrix} 2 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 3 & * & * \\ 0 & 0 & 0 & -5 & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 3 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & -4 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

* = numbers that can be zero or non-zero

In a row echelon form of a matrix:
- → zero rows should be at the bottom
- → Each row has a pivot (a left most non-zero entry)
- → Every pivot is to the right of the pivots on the rows above.
- → Rank of the matrix is the number of pivots.

We can transform the pivots into 1s, by dividing each pivot row by itself.

$\begin{bmatrix} 3 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & -4 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \div 3 \\ \div 1 \\ \div -4 \\ \\ \\ \end{matrix} \longrightarrow \begin{bmatrix} 1 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

REDUCED ROW ECHELON FORM

eg. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & * & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

In a reduced row echelon form matrix :
- → It should already be in a row echelon form
- → Each pivot is 1.
- → Any number above a pivot is 0.
- → Rank of the matrix = number of pivots.

VECTORS

A matrix with a single column is called a vector

eg. $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Vectors can be seen as arrows in the plane or in a higher dimensional space

Components of a vector : magnitude & direction.

NORMS

L1 - norm of a vector $\begin{bmatrix} a \\ b \end{bmatrix} = |(a, b)|_1 = |a| + |b|$

L2 - norm of a vector $\begin{bmatrix} a \\ b \end{bmatrix} = |(a, b)|_2 = \sqrt{a^2 + b^2}$

When we say norm of a vector, we by-default mean the L2-norm. As it is the magnitude of the vector.

The direction of the vector is given by .

$$\theta = \underbrace{\arctan}_{\tan^{-1}}\left(\frac{b}{a}\right)$$

VECTOR SUM & DIFFERENCE

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a-c \\ b-d \end{bmatrix}$$

DISTANCES BETWEEN VECTORS

For vectors $u = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ $v = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

L1 - distance $= |u - v|_1 = |6-1| + |2-5| = |5| + |-3| = 8$

L2 - distance $= |u - v|_2 = \sqrt{(6-1)^2 + (2-5)^2} = \sqrt{5^2 + (-3)^2} = 5.83$

Cosine distance $= \cos\theta$ where θ is the angle between the two vectors

MULTIPLYING A VECTOR BY A SCALAR :

Vector $u = (1, 2)$

$\lambda = 3$

$\therefore 3u = (3, 6)$

vector get stretched by a factor λ.

If $\lambda = -2$

$\therefore -2u = (-2, -4)$

Vector gets stretched and reflected at origin.

DOT PRODUCT

For vectors $u = (2, 4, 1)$ $v = (3, 5, 2)$

Dot product of two vectors u and v is given as

$$\langle u, v \rangle = u \cdot v = \begin{bmatrix} 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = 2 \times 3 + 4 \times 5 + 2 \times 1 = 28$$

Norm of a vector is a dot product of the vector with itself.

$u = (4, 3)$

$$\therefore \begin{bmatrix} 4 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 25$$

L2 - norm $= \sqrt{\text{dot product}(u, u)}$

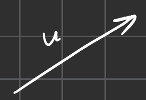
$$\therefore |u|_2 = \sqrt{\langle u, u \rangle}$$

GEOMETRIC DOT PRODUCT

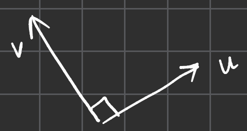
Orthogonal vectors have dot product 0.

$u = (-1, 3) \qquad v = (6, 2)$

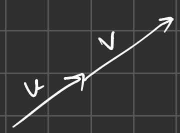
$\therefore \langle u, v \rangle = \begin{bmatrix} 6 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = 0$



$\langle u, u \rangle = |u|^2 = |u| \cdot |u|$



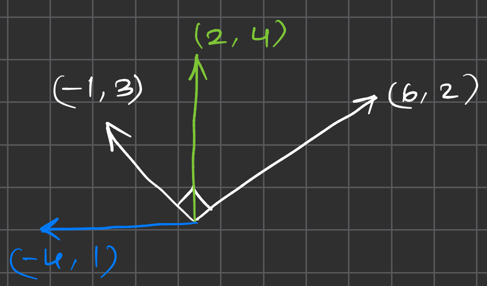
$\langle u, v \rangle = 0$



$\langle u, v \rangle = |u| \; |v|$

$\langle u, v \rangle = |u'| \cdot |v|$

$= |u| |v| \cos \theta$

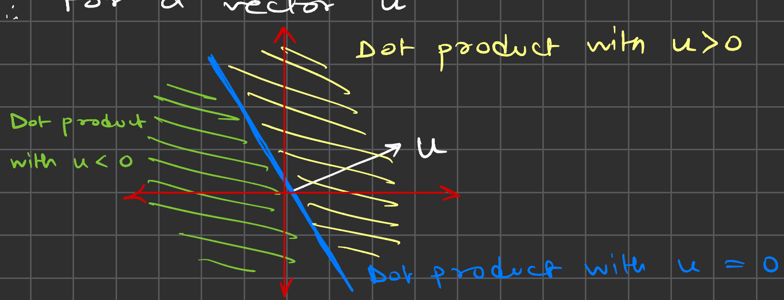


$\Longrightarrow \begin{bmatrix} 6 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 20$ (positive)

$\begin{bmatrix} 6 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = 0$

$\begin{bmatrix} 6 & 2 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \end{bmatrix} = -22$ (negative)

$\therefore$ For a vector u



$\langle u, v \rangle > 0$

$\langle u, v \rangle = 0$

$\langle u, v \rangle < 0$

MULTIPLYING A MATRIX BY A VECTOR :

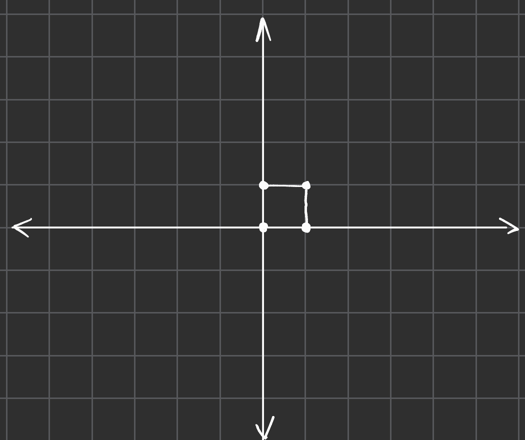
A system of equations is given as,

$$a + b + c = 10$$
$$a + 2b + c = 15$$
$$a + b + 2c = 12$$

$\therefore$ The matrix product is given as,

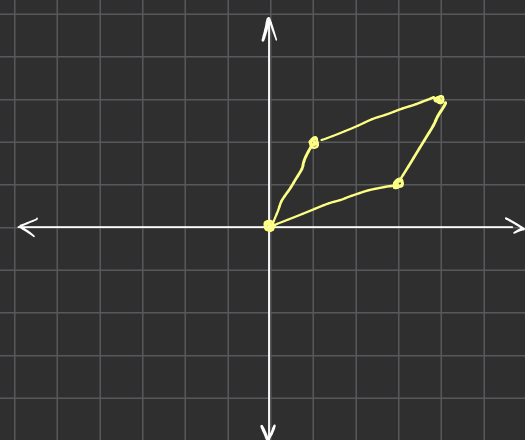
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \\ 12 \end{bmatrix}$$

MATRICES AS LINEAR TRANSFORMATIONS



$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$(0, 0) \rightarrow (0, 0)$
$(1, 0) \rightarrow (3, 1)$
$(0, 1) \rightarrow (1, 2)$
$(1, 1) \rightarrow (4, 3)$



The above matrix transform the square from the initial plane to a parallelogram in the new plane.

LINEAR TRANSFORMATION AS MATRICES

If we know that a basis (a unit square) translates to a given basis, we find the linear transformation between these two planes as follows,

We know that,

$(0, 0) \rightarrow (0, 0)$
$(1, 0) \rightarrow (3, -1)$
$(0, 1) \rightarrow (2, 3)$
$(1, 1) \rightarrow (5, 2)$

these two are called basis vectors.

we only need these two to get the linear transformation between the planes.

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \& \quad \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Longrightarrow \therefore \text{Matrix} = \begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix}$$

MATRIX MULTIPLICATION

Matrix multiplication combines two linear transformations into a third one.

$$\text{Matrix } 1 = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

This matrix transforms a unit square to a parallelogram in a new plane.

$$\text{matrix } 2 = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

This matrix transfooms the unit parallelogram to a new parallelogram in another new plane.

Matrix multiplication is a combination of the above two linear transformation which transforms the unit square to a parallelogram in the third plane.

$$\text{result} = \begin{bmatrix} 5 & 0 \\ 2 & 4 \end{bmatrix}$$

$\therefore$ we can write it as,

all combinations of dot products of the rows and columns

$$\text{result} = \text{Matrix } 2 \times \text{matrix } 1 = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{bmatrix}$$

IDENTITY MATRIX

The identity matrix is the matrix when multiplied by any other matrix gives the same matrix.

eg. $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and so on...

MATRIX INVERSE

The inverse matrix is the matrix for which the product of the matrices is the identity matrix.

inverse matrix

e.g. $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

∴ We can see these as a system of linear equations.

$$\begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = 1$$

$$\begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = 1$$

on solving these, we get

$a = 2/5 \quad b = -1/5 \quad c = -1/5 \quad d = 3/5$

$$\therefore \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2/5 & -1/5 \\ -1/5 & 3/5 \end{bmatrix}$$

WHICH MATRICES DON'T HAVE AN INVERSE?

Non-singular matrices are invertible

Singular matrices are not invertible ie. zero determinant

SINGULARITY AND RANK OF LINEAR TRANSFORMATIONS

As shown earlier, we had given an example of a $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ matrix converting a square from initial plane into a parallelogram in a new plane.

This linear transformation is a non-singular transformation.

If the resulting points after multiplying by a matrix cover the entire plane then the transformation is non-singular and vice versa.

The points that are covering the new plane are called as the image of the transformation.

For example, we have the matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, here the transformation of points (0,0), (0,1), (1,0) and (1,1) results in a line. Thus, when the matrix is singular, the resulting points only cover a small portion of the new plane.

For example, we have the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, here the transformation of any points results in a point – the origin. Thus, this also a singular transformation.

Note :

- the dimension of the image in first example is 2 and the rank of the matrix is also 2.
- in the second example. the dimension of the image is 1 and the rank of the matrix is 1.
- in the last example, the dimension of the image is 0 and the rank of the matrix is 0.

Thus thats another way to calculate rank of a matrix, it is the dimension of the image of the linear transformation.

DETERMINANT AS AN AREA

The determinant of the matrix is the area of the image of the fundamental basis formed by a unit square.

eg. with matrix $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ the determinant is 5.

This is the area of the parallelogram in the new plane.

Simlarly, for a matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, the determinant is 0.

And this is the area of the line in the new plane

For negative determinants, we consider the order of the basis vectors.

example: matrix $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ transforms the vector of cordinates (1,0) to a vector of coordinates (1,2), and the vector of coordinates (0,1) to a vector of coordinates (3,1). It is same as the first example, but in the opposite order.

Thus we say that an area is negative if we take the vectors in counter clockwise order and positive if we take them in clockwise order.

DETERMINANT OF A PRODUCT

$\det(AB) = \det(A) \times \det(B)$

If either of the matrix is singular, then the resulting matrix will also be singular.

DETERMINANT OF INVERSES

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

BASES IN LINEAR ALGEBRA

The main property of a basis is that every point in space can be expressed as linear combination of elements in the basis.

What are non-basis?
Anything that comprises two vectors that either go in the same direction or the opposite direction, as long as they belong to the same line, the two vectors do not form a basis.

SPAN IN LINEAR ALGEBRA

The span of a set of vectors is simply the set of points that can be reached by walking in the direction of these vectors in any combination.

A basis needs to be a minimum spanning set.

Any vector that starts at origin and goes in the same direction is also a basis for that line.

The length of the basis of a space is the dimension of that space.

eg. a line has a basis length of 1 because the line has dimension 1, a plane has a basis length of 2 and the plane has a dimension of 2.

EIGEN VALUES & EIGEN VECTORS

https://www.youtube.com/watch?v=ajXb3N6QEqc

An eigen vector is a vector whose direction remains unchanged after linear transformation. An eigen vector is scaled by a constant factor λ. This is known as the eigen value.

PCA

https://youtu.be/g-Hb26agBFg?si=DyU6qaqm2Afo_heF